

# Nagaoka's theorem in the Holstein-Hubbard model

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## Abstract

Nagaoka's theorem on ferromagnetism in the Hubbard model is extended to the Holstein-Hubbard model. This shows that Nagaoka's ferromagnetism is stable even if the electron-phonon interaction is taken into account. We also prove that Nagaoka's ferromagnetism is stable under the influence of the quantized radiation field.

## 1 Introduction

To build rigorous theory of ferromagnetism is a challenging problem. The Hubbard model is one of the most fundamental model for ferromagnetic metals. Nagaoka constructed a first rigorous example of the ferromagnetism in this model [19]. He proved that the ground state of the model exhibits ferromagnetism when one electron is fewer than half-filling and the Coulomb strength  $U$  is infinitely large. We remark that Thouless also discussed the same mechanism in [26]. Since their discoveries, there have been several crucial developments [6, 8, 25], however, Nagaoka's theorem has been a major milestone in this field. There are several studies concerning Nagaoka's theorem; a generalized version of the theorem was given by Tasaki [23]; Shastry et al. studied the instability of Nagaoka's ferromagnetic state [22], while Kohno extended the theorem to the Hubbard ladders with several holes [4]. This theorem still provides attractive field of studies, see, e.g., [3, 5].

On the one hand, electrons always interact with phonons in actual metals, on the other hand, ferromagnetism is experimentally observed in various metals and has a wide range of uses in daily life. Therefore, if Nagaoka's theorem contains an essence of real ferromagnetism, Nagaoka's ferromagnetism should be stable under the influence of the electron-phonon interaction. The main purpose of this paper is to prove this stability. In addition, we show that stability holds even if the electrons interact with the quantized radiation field. To prove these results, we apply operator theoretic correlation inequalities studied by Miyao in the previous works [9, 10, 11, 12, 13, 14, 15].

We mention some related works. In [6], Lieb gave an example of ferromagnetism in the Hubbard model at half-filling. Recently, Lieb's ferromagnetism is shown to be stable even if the electron-phonon interaction is not so strong [17]. This result is consistent with our results in the present paper.

The organization of the present paper is as follows. In Section 2, we first review Nagaoka's theorem on the Hubbard model. Then we state stability theorems on the

Holstein-Hubbard model and Hubbard model coupled to the quantized radiation field. In Sections 3-6, we prove main theorems. In Appendix A, we prove some correlation inequalities which play an important role in the proofs.

## 2 Results

### 2.1 Nagaoka's theorem revisited

First of all, we review Nagaoka's theorem in the Hubbard model. The Hamiltonian of the Hubbard model is

$$H_H = \sum_{x,y \in \Lambda} \sum_{\sigma \in \{\uparrow, \downarrow\}} t_{xy} c_{x\sigma}^* c_{y\sigma} + U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} + \sum_{\substack{x,y \in \Lambda \\ x \neq y}} U_{xy} n_x n_y. \quad (2.1)$$

$H_H$  acts in the  $N$ -electron Hilbert space  $\mathfrak{E} = \wedge^N(\ell^2(\Lambda) \oplus \ell^2(\Lambda))$ , where  $\wedge^n \mathfrak{h}$  is the  $n$ -fold anti-symmetric tensor product of  $\ell^2(\Lambda) \oplus \ell^2(\Lambda)$ .  $c_{x\sigma}^*$  and  $c_{x\sigma}$  are the standard fermionic operators which create and annihilate the electron at site  $x$  with spin  $\sigma$ .  $n_{x\sigma} = c_{x\sigma}^* c_{x\sigma}$  is the electron number operator at site  $x$  with spin  $\sigma$ . Moreover,  $n_x = n_{x\uparrow} + n_{x\downarrow}$ .  $t_{xy}$  is the hopping matrix element.  $U$  and  $U_{xy}$  are the local and nonlocal Coulomb matrix elements, respectively.

In what follows, we assume the following:

(A. 1)  $\{t_{xy}\}$  and  $\{U_{xy}\}$  are real symmetric matrices.

(A. 2)  $t_{xy} \geq 0$  for all  $x, y \in \Lambda$ .

(A. 3)  $N = |\Lambda| - 1$ .

We derive an effective Hamiltonian describing the system with  $U = \infty$ . Let  $P$  be the Gutzwiller projection given by

$$P = \prod_{x \in \Lambda} (\mathbb{1} - n_{x\uparrow} n_{x\downarrow}), \quad (2.2)$$

where  $\mathbb{1}$  denotes the identity operator.  $P$  is the orthogonal projection onto the subspace with no doubly occupied sites.

**Theorem 2.1** *Assume (A. 1), (A. 2) and (A. 3). We define the effective Hamiltonian  $H_{H,\infty}$  by  $H_{H,\infty} = P H_H^{U=0} P$ , where  $H_H^{U=0}$  is the Hamiltonian  $H_H$  with  $U = 0$ . For all  $z \in \mathbb{C} \setminus \mathbb{R}$ , we have*

$$\lim_{U \rightarrow \infty} (H_H - z)^{-1} = (H_{H,\infty} - z)^{-1} P \quad (2.3)$$

*in the operator norm topology.*

To state Nagaoka's theorem, we need some preparations.

The set of spin configurations with a single hole is denoted by  $\mathcal{S}$ :

$$\mathcal{S} = \left\{ \sigma = \{\sigma_x\}_{x \in \Lambda} \in \{\uparrow, 0, \downarrow\}^N \mid \begin{array}{l} \text{There exists an } x_0 \text{ such that } \sigma_{x_0} = 0 \text{ and } \sigma_x \neq 0 \text{ if } x \neq x_0 \end{array} \right\}. \quad (2.4)$$

We say that the  $x_0$  in (2.4) is the *position of the hole*, and  $\{\sigma_x\}_{x \neq x_0}$  is the *spin configuration of electrons*. For each  $\sigma \in \mathcal{S}$ , the position of the hole is denoted by  $x_0(\sigma)$ .

For each  $\sigma \in \mathcal{S}$ , we denote the number of up spins (resp., down spins) in  $\sigma$  by  $n_\uparrow(\sigma)$  (resp.,  $n_\downarrow(\sigma)$ ), respectively. For each  $M \in \{-(|\Lambda|-1)/2, -(|\Lambda|-2)/2, \dots, (|\Lambda|-1)/2\}$ , we set  $\mathcal{S}_M = \{\sigma \in \mathcal{S} \mid n_\uparrow(\sigma) - n_\downarrow(\sigma) = 2M\}$ .

An element  $(x, \sigma)$  in  $\Lambda \times \mathcal{S}_M$  is called the *hole-spin configuration*, if  $x = x_0(\sigma)$ . In what follows, we denote the set of all hole-spin configurations by  $\mathcal{C}_M$ .

Let  $(x, \sigma) \in \mathcal{C}_M$ . For each  $y \in \Lambda \setminus \{x\}$ , we define a map  $S_{yx} : \mathcal{C}_M \rightarrow \mathcal{C}_M \cup \{\emptyset\}$  by

$$S_{yx}(x, \sigma) = (y, \sigma'), \quad (2.5)$$

where  $\sigma' = \{\sigma'_z\}_{z \in \Lambda} \in \mathcal{S}_M$  is defined by

$$\sigma'_z = \begin{cases} \sigma_y & \text{if } z = x \\ 0 & \text{if } z = y \\ \sigma_z & \text{otherwise} \end{cases} \quad (2.6)$$

and  $S_{yx}(z, \sigma'') = \emptyset$  for all  $(z, \sigma'') \in \mathcal{C}_M$  with  $z \neq x$ .

**Definition 2.2 (Connectivity condition)** We say that  $\Lambda$  has the *connectivity*, if the following condition holds: For every pair of hole-spin configurations  $(x, \sigma), (y, \tau) \in \mathcal{C}_M$ , there exist sites  $x_1, \dots, x_\ell \in \Lambda$  with  $x_1 = x$ ,  $x_\ell = y$  such that

$$t_{x_\ell x_{\ell-1}} t_{x_{\ell-1} x_{\ell-2}} \cdots t_{x_2 x_1} \neq 0 \quad (2.7)$$

and

$$(S_{x_\ell x_{\ell-1}} \circ S_{x_{\ell-1} x_{\ell-2}} \circ \cdots \circ S_{x_2 x_1})(x, \sigma) = (y, \tau). \quad (2.8)$$

The following theorem is due to Tasaki [23].

**Theorem 2.3 (Generalized Nagaoka's theorem)** Assume (A. 1), (A. 2) and (A. 3). Assume that  $\Lambda$  satisfies the connectivity condition. Then the ground state of  $H_{H,\infty}$  has  $S = (|\Lambda| - 1)/2$  and is unique apart from the trivial  $(2S + 1)$ -degeneracy.

**Remark 2.4** In [24], the sufficient condition for the connectivity is given. Using the condition, we know that models with the following (i) and (ii) satisfy the connectivity condition:

- (i)  $\Lambda$  is a triangular, square cubic, fcc, or bcc lattice;
- (ii)  $t_{xy}$  is nonvanishing between nearest neighbor sites.

## 2.2 Stability I: The Holstein-Hubbard model

The Hamiltonian of the Holstein-Hubbard model is given by

$$\begin{aligned} H = & \sum_{x,y \in \Lambda} \sum_{\sigma \in \{\uparrow, \downarrow\}} t_{xy} c_{x\sigma}^* c_{y\sigma} + U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} + \sum_{\substack{x,y \in \Lambda \\ x \neq y}} U_{xy} n_x n_y \\ & + \sum_{x,y \in \Lambda} g_{xy} n_x (b_y^* + b_y) + \sum_{x \in \Lambda} \omega b_x^* b_x. \end{aligned} \quad (2.9)$$

$H$  acts in the Hilbert space  $\mathfrak{E} \otimes \mathfrak{F}$ . Here,  $\mathfrak{F}$  is the bosonic Fock space over  $\ell^2(\Lambda)$  defined by  $\mathfrak{F} = \bigoplus_{n=0}^{\infty} \bigotimes_s^n \ell^2(\Lambda)$ , where  $\bigotimes_s^n \mathfrak{h}$  is the  $n$ -fold symmetric tensor product of  $\mathfrak{h}$ .  $b_x^*$  and  $b_x$  are the bosonic creation- and annihilation operators at site  $x$ . As is well-known,  $H$  is self-adjoint and bounded from below.  $g_{xy}$  is the strength of the electron-phonon interaction. The phonons are assumed to be dispersionless with energy  $\omega > 0$ .  $H$  is a self-adjoint operator, bounded from below.

We assume the following:

**(A. 4)**  $\{g_{xy}\}$  is a real symmetric matrix.

**Theorem 2.5** *Assume (A. 1), (A. 2), (A. 3) and (A. 4). We define the effective Hamiltonian  $H_{\infty}$  by  $H_{\infty} = PH^{U=0}P$ , where  $H^{U=0}$  is the Hamiltonian  $H$  with  $U = 0$ . For all  $z \in \mathbb{C} \setminus \mathbb{R}$ , we have*

$$\lim_{U \rightarrow \infty} (H - z)^{-1} = (H_{\infty} - z)^{-1}P \quad (2.10)$$

in the operator norm topology.

**Theorem 2.6** *Assume (A. 1), (A. 2), (A. 3) and (A. 4). Assume that  $\Lambda$  satisfies the connectivity condition. Then the ground state of  $H_{\infty}$  has  $S = (|\Lambda| - 1)/2$  and is unique apart from the trivial  $(2S + 1)$ -degeneracy.*

### 2.3 Stability II: The Hubbard model coupled to the quantized radiation field

We consider an  $N$ -electron system coupled to the quantized radiation field. Suppose that the lattice  $\Lambda$  is embedded into the region  $V = [-L/2, L/2]^3 \subset \mathbb{R}^3$  with  $L > 0$ . The system is described by the following Hamiltonian:

$$\begin{aligned} \mathbf{H} = & \sum_{\substack{x, y \in \Lambda \\ \sigma = \uparrow, \downarrow}} t_{xy} \exp \left\{ i \int_{C_{xy}} dr \cdot A(r) \right\} c_{x\sigma}^* c_{y\sigma} + \sum_{k \in V^*} \sum_{\lambda=1,2} \omega(k) a(k, \lambda)^* a(k, \lambda) \\ & + U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} + \sum_{\substack{x, y \in \Lambda \\ x \neq y}} U_{xy} n_x n_y. \end{aligned} \quad (2.11)$$

$\mathbf{H}$  acts in the Hilbert space  $\mathfrak{E} \otimes \mathfrak{R}$ .  $\mathfrak{R}$  is the Fock space over  $\ell^2(V^* \times \{1, 2\})$  with  $V^* = (\frac{2\pi}{L}\mathbb{Z})^3$ , namely,  $\mathfrak{R} = \bigoplus_{n \geq 0} \bigotimes_s^n \ell^2(V^* \times \{1, 2\})$ .  $a(k, \lambda)^*$  and  $a(k, \lambda)$  are the bosonic creation- and annihilation operators, respectively. These operators satisfy the following commutation relations:

$$[a(k, \lambda), a(k', \lambda')^*] = \delta_{\lambda\lambda'} \delta_{kk'}, \quad [a(k, \lambda), a(k', \lambda')] = 0. \quad (2.12)$$

The quantized vector potential is given by

$$A(x) = |V|^{-1/2} \sum_{k \in V^*} \sum_{\lambda=1,2} \frac{\chi_{\kappa}(k)}{\sqrt{2\omega(k)}} \varepsilon(k, \lambda) \left( e^{ik \cdot x} a(k, \lambda) + e^{-ik \cdot x} a(k, \lambda)^* \right). \quad (2.13)$$

The form factor  $\chi_{\kappa}$  is the indicator function of the ball of radius  $0 < \kappa < \infty$ . The dispersion relation  $\omega(k)$  is chosen to be  $\omega(k) = |k|$  for  $k \in V^* \setminus \{0\}$ ,  $\omega(0) = m_0$  with

$0 < m_0 < \infty$ .  $C_{xy}$  is a piecewise smooth curve from  $x$  to  $y$ . For concreteness, the polarization vectors are chosen as

$$\varepsilon(k, 1) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \varepsilon(k, 2) = \frac{k}{|k|} \wedge \varepsilon(k, 1). \quad (2.14)$$

To avoid ambiguity, we set  $\varepsilon(k, \lambda) = 0$  if  $k_1 = k_2 = 0$ .  $A(x)$  is essentially self-adjoint. We denote its closure by the same symbol. This model was introduced by Giuliani at el. in [2]. Remark that  $H$  is essentially self-adjoint and bounded from below. We denote its closure by the same symbol.

**Remark 2.7** A more precise definition of  $\int_{C_{xy}} dr \cdot A(r)$  is given in Section 6.  $\diamond$

**Theorem 2.8** Assume (A. 1), (A. 2) and (A. 3). We define the effective Hamiltonian  $H_\infty$  by  $H_\infty = PH^{U=0}P$ , where  $H^{U=0}$  is the Hamiltonian  $H$  with  $U = 0$ . For all  $z \in \mathbb{C} \setminus \mathbb{R}$ , we have

$$\lim_{U \rightarrow \infty} (H - z)^{-1} = (H_\infty - z)^{-1}P \quad (2.15)$$

in the operator norm topology.

**Theorem 2.9** Assume (A. 1), (A. 2) and (A. 3). Assume that  $\Lambda$  satisfies the connectivity condition. Then the ground state of  $H_\infty$  has  $S = (|\Lambda| - 1)/2$  and is unique apart from the trivial  $(2S + 1)$ -degeneracy.

### 3 Proof of Theorem 2.5

Let  $H_1 = P^\perp H P^\perp$  and let  $H_{01} = PHP^\perp + P^\perp H P$ . Using the fact that  $P(\sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow})P = 0$ , we have  $H_\infty = PHP$ . Accordingly, we have  $H = H_\infty + H_1 + H_{01}$ . Moreover, since  $P$  commutes with boson operators, we have

$$H_{01} = PTP^\perp + P^\perp TP, \quad (3.1)$$

where  $T = \sum_{x, y \in \Lambda} \sum_{\sigma \in \{\uparrow, \downarrow\}} t_{xy} c_{x\sigma}^* c_{y\sigma}$ . In particular,  $H_{01}$  is a bounded operator.

**Lemma 3.1** Let  $E(H_1) = \min \text{spec}(H_1)$ , where  $\text{spec}(H_1)$  is spectrum of  $H_1$ . Then

$$E(H_1) \geq C + U, \quad (3.2)$$

where  $C$  is a constant independent of  $U$ .

*Proof.* For any self-adjoint operator  $A$ , bounded from below, we set  $E(A) := \inf \text{spec}(A)$ . We divide  $H_1$  into two parts as  $H_1 = H_{1,U=0} + U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} P^\perp$ , where  $H_{1,U=0}$  is the Hamiltonian  $H_1$  with  $U = 0$ . We have  $E(H_1) \geq E(H_{1,U=0}) + E(U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} P^\perp)$ . Since  $E(U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} P^\perp) \geq U$ , we obtain the desired result.  $\square$

**Corollary 3.2** If  $U$  is sufficiently large, then  $H_1^{-1} P^\perp$  is a bounded operator.

**Lemma 3.3** *Let  $z \in \mathbb{C} \setminus \mathbb{R}$ . We have*

$$\|(H_1 - z)^{-1}P^\perp\| \leq \{E(H_1) - |z|\}^{-1}, \quad (3.3)$$

*provided that  $U$  is sufficiently large.*

*Proof.* Since  $H_1^{-1}P^\perp$  is bounded, we have

$$(H_1 - z)^{-1}P^\perp = \sum_{n=0}^{\infty} (H_1^{-1}P^\perp z)^n H_1^{-1}P^\perp. \quad (3.4)$$

Thus, we have  $\|(H_1 - z)^{-1}P^\perp\| \leq \sum_{n=0}^{\infty} E(H_1)^{-n-1}|z|^n = \{E(H_1) - |z|\}^{-1}$ .  $\square$

**Lemma 3.4** *Let  $z \in \mathbb{C} \setminus \mathbb{R}$ . If  $|\operatorname{Im} z|$  is sufficiently large, then we have*

$$\lim_{U \rightarrow \infty} \left\| (H - z)^{-1} - (H_\infty + H_1 - z)^{-1} \right\| = 0. \quad (3.5)$$

*Proof.* First, remark that

$$\left\{ (H - z)^{-1} - (H_\infty + H_1 - z)^{-1} \right\} P = (H - z)^{-1} P^\perp (-H_{01}) (H_\infty - z)^{-1} P. \quad (3.6)$$

The norm of  $(H - z)^{-1}P^\perp$  is estimated as follows: Since

$$(H - z)^{-1}P^\perp = \sum_{n=0}^{\infty} (-1)^n \left\{ (H_\infty + H_1 - z)^{-1} H_{01} \right\}^n (H_1 - z)^{-1} P^\perp, \quad (3.7)$$

we have, by Lemma 3.3,

$$\begin{aligned} \|(H - z)^{-1}P^\perp\| &\leq \left( \sum_{n=0}^{\infty} |\operatorname{Im} z|^{-n} \|H_{01}\|^n \right) \{E(H_1) - |z|\}^{-1} \\ &:= C_z \{E(H_1) - |z|\}^{-1}. \end{aligned} \quad (3.8)$$

Thus, by Lemma 3.1 and (3.6),

$$\left\| \left\{ (H - z)^{-1} - (H_\infty + H_1 - z)^{-1} \right\} P \right\| \leq C_z \|H_{01}\| |\operatorname{Im} z|^{-1} \{E(H_1) - |z|\}^{-1} \rightarrow 0 \quad (3.9)$$

as  $U \rightarrow \infty$ .

Next, since

$$\left\{ (H - z)^{-1} - (H_\infty + H_1 - z)^{-1} \right\} P^\perp = (H - z)^{-1} P (-H_{01}) (H_1 - z)^{-1} P^\perp, \quad (3.10)$$

we obtain, by Lemmas 3.1 and 3.3,

$$\left\| \left\{ (H - z)^{-1} - (H_\infty + H_1 - z)^{-1} \right\} P^\perp \right\| \leq |\operatorname{Im} z|^{-1} \|H_{01}\| \{E(H_1) - |z|\}^{-1} \rightarrow 0 \quad (3.11)$$

as  $U \rightarrow \infty$ .  $\square$

**Lemma 3.5** *Let  $z \in \mathbb{C} \setminus \mathbb{R}$ . If  $|\operatorname{Im} z|$  is sufficiently large, then we have*

$$\lim_{U \rightarrow \infty} \left\| (H_\infty + H_1 - z)^{-1} - (H_\infty - z)^{-1} P \right\| = 0. \quad (3.12)$$

*Proof.* Remark that

$$(H_\infty + H_1 - z)^{-1} - (H_\infty - z)^{-1} P = (H_1 - z)^{-1} P^\perp. \quad (3.13)$$

Hence, we obtain, by Lemmas 3.1 and 3.3,

$$\left\| (H_\infty + H_1 - z)^{-1} - (H_\infty - z)^{-1} P \right\| \leq \{E(H_1) - |z|\}^{-1} \rightarrow 0 \quad (3.14)$$

as  $U \rightarrow \infty$ .  $\square$

*Completion of proof of Theorem 2.5*

By Lemmas 3.4, 3.5 and [20, Theorem VIII. 19], we obtain the desired result in the theorem.  $\square$

## 4 Operator theoretic correlation inequalities

### 4.1 Self-dual cones

In this section, we will review a general theory of correlation inequalities developed in [9, 10, 11, 12, 13, 14, 15, 16].

Let  $\mathfrak{H}$  be a complex Hilbert space. By a *convex cone*, we understand a closed convex set  $\mathfrak{P} \subset \mathfrak{H}$  such that  $t\mathfrak{P} \subseteq \mathfrak{P}$  for all  $t \geq 0$  and  $\mathfrak{P} \cap (-\mathfrak{P}) = \{0\}$ . In what follows, we always assume that  $\mathfrak{P} \neq \{0\}$ .

**Definition 4.1** The *dual cone* of  $\mathfrak{P}$  is defined by

$$\mathfrak{P}^\dagger = \{\eta \in \mathfrak{H} \mid \langle \eta | \xi \rangle \geq 0 \ \forall \xi \in \mathfrak{P}\}. \quad (4.1)$$

We say that  $\mathfrak{P}$  is *self-dual* if  $\mathfrak{P} = \mathfrak{P}^\dagger$ .  $\diamond$

**Remark 4.2** [16]  $\mathfrak{P}$  is a self-dual cone if and only if  $\mathfrak{P}$  is a Hilbert cone.<sup>1</sup>  $\diamond$

**Definition 4.3** (i) A vector  $\xi$  is said to be *positive w.r.t.  $\mathfrak{P}$*  if  $\xi \in \mathfrak{P}$ . We write this as  $\xi \geq 0$  w.r.t.  $\mathfrak{P}$ .

(ii) A vector  $\eta \in \mathfrak{P}$  is called *strictly positive w.r.t.  $\mathfrak{P}$*  whenever  $\langle \xi | \eta \rangle > 0$  for all  $\xi \in \mathfrak{P} \setminus \{0\}$ . We write this as  $\eta > 0$  w.r.t.  $\mathfrak{P}$ .  $\diamond$

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<sup>1</sup> Let  $\mathfrak{H}$  be a complex Hilbert space. A convex cone  $\mathfrak{P}$  in  $\mathfrak{H}$  is called a *Hilbert cone*, if it satisfies the following: (i)  $\langle \xi | \eta \rangle \geq 0$  for all  $\xi, \eta \in \mathfrak{P}$ ; (ii) Let  $\mathfrak{H}_\mathbb{R}$  be a real subspace of  $\mathfrak{H}$  generated by  $\mathfrak{P}$ . Then for all  $\xi \in \mathfrak{H}_\mathbb{R}$ , there exist  $\xi_+, \xi_- \in \mathfrak{P}$  such that  $\xi = \xi_+ - \xi_-$  and  $\langle \xi_+ | \xi_- \rangle = 0$ ; (iii)  $\mathfrak{H} = \mathfrak{H}_\mathbb{R} + i\mathfrak{H}_\mathbb{R} = \{\xi + i\eta \mid \xi, \eta \in \mathfrak{H}_\mathbb{R}\}$ .

**Example 1** Let  $\mathfrak{X}$  be a complex Hilbert space and let  $\{x_n\}_{n \in \mathbb{N}}$  be a complete orthonormal system in  $\mathfrak{X}$ . We set

$$\mathfrak{P} = \overline{\text{Coni}\{x_n \mid n \in \mathbb{N}\}}, \quad (4.2)$$

where  $\text{Coni}(S)$  is the conical hull of  $S$  and  $\overline{\text{Coni}(S)}$  indicates the closure of  $\text{Coni}(S)$ . Then  $\mathfrak{P}$  is a self-dual cone in  $\mathfrak{X}$ .  $x \geq 0$  w.r.t.  $\mathfrak{P}$  if and only if  $\langle x_n | x \rangle \geq 0$  for all  $n \in \mathbb{N}$ . In addition,  $x > 0$  w.r.t.  $\mathfrak{P}$  if and only if  $\langle x_n | x \rangle > 0$  for all  $n \in \mathbb{N}$ .  $\diamond$

**Example 2** Let  $(M, \mu)$  be a  $\sigma$ -finite measure space. We set

$$L^2(M, d\mu)_+ = \{f \in L^2(M, d\mu) \mid f(m) \geq 0 \text{ } \mu\text{-a.e.}\}. \quad (4.3)$$

$L^2(M, d\mu)_+$  is a self-dual cone in  $L^2(M, d\mu)$ .  $f \geq 0$  w.r.t.  $L^2(M, d\mu)_+$  if and only if  $f(m) \geq 0$   $\mu$ -a.e.. On the other hand,  $f > 0$  w.r.t.  $L^2(M, d\mu)_+$  if and only if  $f(m) > 0$   $\mu$ -a.e..  $\diamond$

## 4.2 Operator inequalities associated with self-dual cones

In subsequent sections, we use the following operator inequalities.

**Definition 4.4** We denote by  $\mathcal{B}(\mathfrak{H})$  the set of all bounded linear operators on  $\mathfrak{H}$ . Let  $A, B \in \mathcal{B}(\mathfrak{H})$ . Let  $\mathfrak{P}$  be a self-dual cone in  $\mathfrak{H}$ .

If  $A\mathfrak{P} \subseteq \mathfrak{P}$ ,<sup>2</sup> we then write this as  $A \geq 0$  w.r.t.  $\mathfrak{P}$ .<sup>3</sup> In this case, we say that  $A$  *preserves the positivity w.r.t.  $\mathfrak{P}$* . Let  $\mathfrak{H}_{\mathbb{R}}$  be a real subspace generated by  $\mathfrak{P}$ . Suppose that  $A\mathfrak{H}_{\mathbb{R}} \subseteq \mathfrak{H}_{\mathbb{R}}$  and  $B\mathfrak{H}_{\mathbb{R}} \subseteq \mathfrak{H}_{\mathbb{R}}$ . If  $(A - B)\mathfrak{P} \subseteq \mathfrak{P}$ , then we write this as  $A \geq B$  w.r.t.  $\mathfrak{P}$ .  $\diamond$

**Remark 4.5**  $A \geq 0$  w.r.t.  $\mathfrak{P} \iff \langle \xi | A\eta \rangle \geq 0$  for all  $\xi, \eta \in \mathfrak{P}$ .  $\diamond$

The following proposition is fundamental in the present paper.

**Proposition 4.6** Let  $A, B, C, D \in \mathcal{B}(\mathfrak{H})$  and let  $a, b \in \mathbb{R}$ .

- (i) If  $A \geq 0, B \geq 0$  w.r.t.  $\mathfrak{P}$  and  $a, b \geq 0$ , then  $aA + bB \geq 0$  w.r.t.  $\mathfrak{P}$ .
- (ii) If  $A \geq B \geq 0$  and  $C \geq D \geq 0$  w.r.t.  $\mathfrak{P}$ , then  $AC \geq BD \geq 0$  w.r.t.  $\mathfrak{P}$ .
- (iii) If  $A \geq 0$  w.r.t.  $\mathfrak{P}$ , then  $A^* \geq 0$  w.r.t.  $\mathfrak{P}$ .

*Proof.* (i) is trivial.

(ii) If  $X \geq 0$  and  $Y \geq 0$  w.r.t.  $\mathfrak{P}$ , we have  $XY\mathfrak{P} \subseteq X\mathfrak{P} \subseteq \mathfrak{P}$ . Hence, it holds that  $XY \geq 0$  w.r.t.  $\mathfrak{P}$ . Hence, we have

$$AC - BD = \underbrace{A}_{\geq 0} \underbrace{(C - D)}_{\geq 0} + \underbrace{(A - B)}_{\geq 0} \underbrace{D}_{\geq 0} \geq 0 \text{ w.r.t. } \mathfrak{P}.$$

<sup>2</sup> For each subset  $\mathfrak{C} \subseteq \mathfrak{H}$ ,  $A\mathfrak{C}$  is defined by  $A\mathfrak{C} = \{Ax \mid x \in \mathfrak{C}\}$ .

<sup>3</sup>This symbol was introduced by Miura [18].



(iii) For each  $\xi, \eta \in \mathfrak{P}$ , we know that

$$\langle \xi | A^* \eta \rangle = \underbrace{\langle A}_{\geq 0} \underbrace{\xi}_{\geq 0} | \underbrace{\eta}_{\geq 0} \rangle \geq 0. \quad (4.4)$$

Thus, by Remark 4.5, we conclude (iii).  $\square$

**Proposition 4.7** *Let  $\{A_n\}_{n=1}^\infty \subseteq \mathcal{B}(\mathfrak{H})$  and let  $A \in \mathcal{B}(\mathfrak{H})$ . Suppose that  $A_n$  converges to  $A$  in the weak operator topology. If  $A_n \geq 0$  w.r.t.  $\mathfrak{P}$  for all  $n \in \mathbb{N}$ , then  $A \geq 0$  w.r.t.  $\mathfrak{P}$ .*

*Proof.* By Remark 4.5,  $\langle \xi | A_n \eta \rangle \geq 0$  for all  $\xi, \eta \in \mathfrak{P}$ . Thus,  $\langle \xi | A \eta \rangle = \lim_{n \rightarrow \infty} \langle \xi | A_n \eta \rangle \geq 0$  for all  $\xi, \eta \in \mathfrak{P}$ . By Remark 4.5 again, we conclude that  $A \geq 0$  w.r.t.  $\mathfrak{P}$ .  $\square$

**Definition 4.8** Let  $A \in \mathcal{B}(\mathfrak{H})$ . We write  $A \triangleright 0$  w.r.t.  $\mathfrak{P}$ , if  $A\xi > 0$  w.r.t.  $\mathfrak{P}$  for all  $\xi \in \mathfrak{P} \setminus \{0\}$ . In this case, we say that  $A$  improves the positivity w.r.t.  $\mathfrak{P}$ .  $\diamond$

The following theorem plays an important role.

**Theorem 4.9** (Perron–Frobenius–Faris) *Let  $A$  be a self-adjoint operator, bounded from below. We set  $E(A) := \inf \text{spec}(A)$ . Suppose that  $0 \leq e^{-tA}$  w.r.t.  $\mathfrak{P}$  for all  $t \geq 0$  and that  $E(A)$  is an eigenvalue. Let  $P_A$  be the orthogonal projection onto the closed subspace spanned by eigenvectors associated with  $E(A)$ . Then, the following are equivalent:*

- (i)  $\dim \text{ran } P_A = 1$  and  $P_A \triangleright 0$  w.r.t.  $\mathfrak{P}$ .
- (ii)  $0 \triangleleft e^{-tA}$  w.r.t.  $\mathfrak{P}$  for all  $t > 0$ .
- (iii) For each  $\xi, \eta \in \mathfrak{P} \setminus \{0\}$ , there exists a  $t > 0$  such that  $\langle \xi | e^{-tA} \eta \rangle > 0$ .

*Proof.* See, e.g., [1, 9, 21].  $\square$

**Remark 4.10** (i) is equivalent to the following: there exists a unique  $\xi \in \mathfrak{H}$  such that  $\xi > 0$  w.r.t.  $\mathfrak{P}$  and  $P_A = |\xi\rangle\langle\xi|$ . Of course,  $\xi$  satisfies  $A\xi = E(A)\xi$ .  $\diamond$

## 5 Proof of Theorem 2.6

### 5.1 The $S^{(3)} = M$ subspace

Let  $S^{(3)} = \frac{1}{2} \sum_{x \in \Lambda} (n_{x\uparrow} - n_{x\downarrow})$ . We denote the spectrum of  $S^{(3)}$  by  $\text{spec}(S^{(3)})$ . Remark that  $\text{spec}(S^{(3)}) = \{-(|\Lambda| - 1)/2, -(|\Lambda| - 2)/2, \dots, (|\Lambda| - 1)/2\}$ . Thus, we have the following decomposition:

$$\mathfrak{E} = \bigoplus_{M \in \text{spec}(S^{(3)})} \mathfrak{E}(M), \quad \mathfrak{E}(M) = \ker(S^{(3)} - M). \quad (5.1)$$

In this paper, we call  $\mathfrak{E}(M) \otimes \mathfrak{F}$  the  $S^{(3)} = M$  subspace.

## 5.2 The Lang-Firsov transformation

Let

$$L = \omega^{-1} \sum_{x,y \in \Lambda} g_{xy} n_x (b_y^* - b_y). \quad (5.2)$$

$L$  is essentially anti-self-adjoint. We denote its closure by  $L$ . We set  $H'_\infty = e^L H_\infty e^{-L}$ .

Let

$$\mathbf{T} = \sum_{x,y \in \Lambda} \sum_{\sigma=\uparrow,\downarrow} \vartheta_{xy} T_{xy}(\sigma), \quad (5.3)$$

where

$$T_{xy}(\sigma) = t_{xy} P c_{x\sigma}^* c_{y\sigma} P \quad (5.4)$$

and

$$\vartheta_{xy} = \exp \left\{ -i\sqrt{2}\omega^{-3/2} \sum_{z \in \Lambda} (g_{xz} - g_{yz}) p_z \right\} \quad (5.5)$$

with  $p_x$  the self-adjoint extension of  $i\sqrt{\frac{\omega}{2}}(b_x^* - b_x)$ .

Since  $P$  commutes with  $L$ , we obtain

$$H'_\infty = \mathbf{T} + U_{\text{eff}} P + \omega N_b P + \text{Const.}, \quad (5.6)$$

where  $N_b = \sum_{x \in \Lambda} b_x^* b_x$  and

$$U_{\text{eff}} = \sum_{x \neq y} U_{\text{eff},xy} n_x n_y \quad (5.7)$$

with  $U_{\text{eff},xy} = U_{xy} - \omega^{-1} \sum_{z \in \Lambda} g_{xz} g_{zy}$ . Henceforth, we ignore the constant term in (5.6), because it does not affect our proof below.

## 5.3 Definition of the connector and its related operators

For each  $(x, \boldsymbol{\sigma}) \in \mathcal{C}_M$ , we define  $\boldsymbol{\sigma}' = \{\sigma'_z\}_{z \in \Lambda} \in \{\uparrow, \downarrow\}^\Lambda$  by

$$\sigma'_z = \begin{cases} \uparrow & \text{if } z = x \\ \sigma_z & \text{otherwise.} \end{cases} \quad (5.8)$$

Following Tasaki [23, 24], we define a complete orthonormal system  $\{|x, \boldsymbol{\sigma}\rangle \mid (x, \boldsymbol{\sigma}) \in \mathcal{C}_M\}$  by

$$|x, \boldsymbol{\sigma}\rangle = c_{x\uparrow} \prod'_{z \in \Lambda} c_{z\sigma'_z}^* \Omega_f, \quad (5.9)$$

where  $\Omega_f$  is the Fock vacuum and  $\prod'_{z \in \Lambda}$  indicates the ordered product according to an arbitrarily fixed order in  $\Lambda$ .

For each  $M \in \{-(|\Lambda| - 1)/2, -(|\Lambda| - 2)/2, \dots, (|\Lambda| - 1)/2\}$ , a canonical self-dual cone in  $\mathfrak{E}(M)$  is defined by

$$\mathfrak{E}_+(M) = \text{Coni} \left\{ |x, \boldsymbol{\sigma}\rangle \mid (x, \boldsymbol{\sigma}) \in \mathcal{C}_M \right\}. \quad (5.10)$$

We begin with the following lemma.

**Lemma 5.1** For every  $x, y \in \Lambda$  and  $\sigma \in \{\uparrow, \downarrow\}$ , we have

$$\langle y', \sigma | \{-T_{xy}(\sigma)\} | x', \tau \rangle = t_{xy} \delta_{xx'} \delta_{yy'} \delta_{S_{yx}(\tau) \sigma}. \quad (5.11)$$

In additon,  $-T_{xy}(\sigma) \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$ .

*Proof.* To check (5.11) is easy, or see [24, Lemma 4.2]. Let  $\phi, \psi \in \mathfrak{E}_+(M)$ . Thus, we can express these as  $\phi = \sum_{(x, \sigma) \in \mathcal{C}_M} \phi_{x, \sigma} |x, \sigma\rangle$ ,  $\psi = \sum_{(x, \sigma) \in \mathcal{C}_M} \psi_{x, \sigma} |x, \sigma\rangle$  with  $\phi_{x, \sigma} \geq 0$  and  $\psi_{x, \sigma} \geq 0$ . Using these expressions and (5.11), we can check that  $\langle \phi | \{-T_{xy}(\sigma)\} \psi \rangle \geq 0$ . Thus, by Remark 4.5, we conclude that  $-T_{xy}(\sigma) \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$ .  $\square$

**Definition 5.2** We say that  $p = \{x_1, x_2, \dots, x_\ell\}$  in Definition 2.2 is a *connector* between  $(x, \sigma)$  and  $(y, \tau)$ . The number  $\ell - 1$  is called the *length of  $p$*  and denoted by  $|p|$ .  $\diamond$

For each connector  $p$  between  $(x, \sigma)$  and  $(y, \tau)$ , we define a linear operator  $\tau(p)$  by

$$\tau(p) = \{-T_{x_{\ell-1}x_\ell}(\sigma_\ell)\} \{-T_{x_{\ell-2}x_{\ell-1}}(\sigma_{\ell-1})\} \cdots \{-T_{x_1x_2}(\sigma_1)\}. \quad (5.12)$$

**Proposition 5.3** Let  $(x, \sigma), (y, \tau) \in \mathcal{C}_M$ . Let  $p$  be a connector between  $(x, \sigma)$  and  $(y, \tau)$ . Then we have

$$\tau(p) \geq t_{x_\ell x_{\ell-1}} \cdots t_{x_2 x_1} |y, \tau\rangle \langle x, \sigma| \quad \text{w.r.t. } \mathfrak{E}_+(M). \quad (5.13)$$

In particular, we have  $\langle y, \tau | \tau(p) | x, \sigma \rangle \geq t_{x_\ell x_{\ell-1}} \cdots t_{x_2 x_1} > 0$ .

*Proof.* We set

$$(x_1, \sigma_1) := (x, \sigma), (x_2, \sigma_2) := S_{x_2 x_1}(x_1, \sigma_1), \dots, (x_\ell, \sigma_\ell) := S_{x_\ell x_{\ell-1}}(x_{\ell-1}, \sigma_{\ell-1}). \quad (5.14)$$

Remark that  $(x_\ell, \sigma_\ell) = (y, \tau)$ . Define  $P_1 = |x_1, \sigma_1\rangle \langle x_1, \sigma_1|, \dots, P_\ell = |x_\ell, \sigma_\ell\rangle \langle x_\ell, \sigma_\ell|$ . Since  $\mathbb{1} \geq P_k$  w.r.t.  $\mathfrak{E}_+(M)$  for all  $k = 1, \dots, \ell$ ,<sup>4</sup> we obtain, by Proposition 4.6 and Lemma 5.1,

$$\begin{aligned} \tau(p) &= \mathbb{1} \{-T_{x_{\ell-1}x_\ell}(\sigma_\ell)\} \mathbb{1} \{-T_{x_{\ell-2}x_{\ell-1}}(\sigma_{\ell-1})\} \mathbb{1} \cdots \mathbb{1} \{-T_{x_1x_2}(\sigma_1)\} \mathbb{1} \\ &\geq P_\ell \{-T_{x_{\ell-1}x_\ell}(\sigma_\ell)\} P_{\ell-1} \{-T_{x_{\ell-2}x_{\ell-1}}(\sigma_{\ell-1})\} P_{\ell-2} \cdots P_2 \{-T_{x_1x_2}(\sigma_1)\} P_1 \\ &\geq t_{x_\ell x_{\ell-1}} \cdots t_{x_2 x_1} |y, \tau\rangle \langle x, \sigma|. \end{aligned} \quad (5.15)$$

This completes the proof.  $\square$

**Lemma 5.4** For each  $\beta \geq 0$ ,  $e^{-\beta U_{\text{eff}} P} \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$  for all  $\beta \geq 0$ .

*Proof.* For each  $(x, \sigma) \in \mathcal{C}_M$ ,  $|x, \sigma\rangle$  is an eigenvector of  $U_{\text{eff}}$ . We denote the corresponding eigenvalue by  $U_{\text{eff}}(x, \sigma)$ . Then one sees that  $e^{-\beta U_{\text{eff}}} |x, \sigma\rangle = e^{-\beta U_{\text{eff}}(x, \sigma)} |x, \sigma\rangle$ . Since  $e^{-\beta U_{\text{eff}}(x, \sigma)}$  is a positive number, we conclude the assertion by Remark 4.5.  $\square$

<sup>4</sup> To show this, remark that  $\mathbb{1} = \sum_{(x, \sigma) \in \mathcal{C}_M} |x, \sigma\rangle \langle x, \sigma|$ . Since  $|x, \sigma\rangle \langle x, \sigma| \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$  for all  $(x, \sigma) \in \mathcal{C}_M$ , we obtain that  $\mathbb{1} \geq |x, \sigma\rangle \langle x, \sigma|$  w.r.t.  $\mathfrak{E}_+(M)$  for all  $(x, \sigma) \in \mathcal{C}_M$ .

## 5.4 The Schrödinger representation

Let  $q_x = \frac{1}{\sqrt{2\omega}}(b_x + b_x^*)$ .  $q_x$  is essentially self-adjoint. We denote its closure by the same symbol. Recall that  $p_x$  is the closure of  $i\sqrt{\frac{\omega}{2}}(b_x^* - b_x)$ .

The bosonic Fock space  $\mathfrak{F}$  can be naturally identified with  $L^2(Q)$  with  $Q = \mathbb{R}^{|\Lambda|}$ . In addition,  $q_x$  and  $p_x$  can be identified with a multiplication operator and  $p_x = -i\partial/\partial q_x$ , respectively. In what follows, we use this representation. A natural self-dual cone in  $L^2(Q)$  is

$$L^2(Q)_+ = \{f \in L^2(Q) \mid f(\mathbf{q}) \geq 0 \text{ a.e. } \mathbf{q}\}. \quad (5.16)$$

**Lemma 5.5** *We have the following:*

- (i) *For each  $x, y \in \Lambda$ ,  $\vartheta_{xy} \geq 0$  w.r.t.  $L^2(Q)_+$ .*
- (ii)  *$e^{-\beta\omega N_b} \triangleright 0$  w.r.t.  $L^2(Q)_+$  for all  $\beta > 0$ .*

*Proof.* (i) Since  $p_x = -i\partial/\partial q_x$ ,  $e^{iap_x}$  is a translation operator. Thus,  $e^{iap_x} \geq 0$  w.r.t.  $L^2(Q)_+$  for all  $a \in \mathbb{R}$ , which implies  $\theta_{xy} \geq 0$  w.r.t.  $L^2(Q)_+$  for all  $x, y \in \Lambda$ .

(ii) We remark that  $\omega N_b = \frac{1}{2} \sum_{x \in \Lambda} (-\nabla_{q_x}^2 + q_x^2) - \frac{|\Lambda|}{2}$ . Hence, (ii) follows from [21, Theorems XIII. 44 and XIII. 47].  $\square$

## 5.5 A natural self-dual cone in $\mathfrak{E}(M) \otimes \mathfrak{F}$

First, we remark the following identification:

$$\mathfrak{E}(M) \otimes L^2(Q) = \int_Q^\oplus \mathfrak{E}(M) d\mathbf{q}. \quad (5.17)$$

We take the following self-dual cone in the full space  $\mathfrak{E}(M) \otimes \mathfrak{F}$ :

$$\mathfrak{P}_M = \left\{ \Phi \in \mathfrak{E}(M) \otimes L^2(Q) \mid \Phi(\mathbf{q}) \geq 0 \text{ w.r.t. } \mathfrak{E}_+(M) \text{ for a.e. } \mathbf{q} \right\}. \quad (5.18)$$

**Lemma 5.6**  *$\mathfrak{P}_M$  is a self-dual cone in  $\mathfrak{E}(M) \otimes L^2(Q)$ .*

*Proof.* We will prove that  $\mathfrak{P}_M^\dagger = \mathfrak{P}_M$ . It is easy to check that  $\mathfrak{P}_M \subseteq \mathfrak{P}_M^\dagger$ . Thus, it suffices to show the converse. Let  $\Psi = \int_Q^\oplus \Psi(\mathbf{q}) d\mathbf{q} \in \mathfrak{P}_M^\dagger$ . Then, for all  $\Phi \in \mathfrak{P}_M$ , we have  $\langle \Psi | \Phi \rangle \geq 0$ . By choosing  $\Phi$  as  $\Phi = |x, \sigma\rangle \otimes f$  with  $f \in L^2(Q)_+$ , we have  $\int_Q \langle \Psi(\mathbf{q}) | x, \sigma \rangle f(\mathbf{q}) d\mathbf{q} \geq 0$ . Since  $f$  is arbitrarily, we have  $\langle \Psi(\mathbf{q}) | x, \sigma \rangle \geq 0$  for all  $(x, \sigma) \in \mathcal{C}_M$ , which implies that  $\psi(\mathbf{q}) \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$ . Hence,  $\Psi \in \mathfrak{P}_M$ .  $\square$

**Lemma 5.7** *Let  $A \in \mathcal{B}(\mathfrak{E}(M))$  and  $B \in \mathcal{B}(L^2(Q))$ . Assume that  $A \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$  and  $B \geq 0$  w.r.t.  $L^2(Q)_+$ . We have  $A \otimes \mathbb{1} \geq 0$ ,  $\mathbb{1} \otimes B \geq 0$  and  $A \otimes B \geq 0$  w.r.t.  $\mathfrak{P}_M$ .*

*Proof.* Let  $\Psi = \int_Q^\oplus \Psi(\mathbf{q}) d\mathbf{q} \in \mathfrak{P}_M$ . Thus,  $\Psi(\mathbf{q}) \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$  for a.e.  $\mathbf{q}$ . Remark that

$$A \otimes \mathbb{1} \Psi = \int_Q^\oplus A \Psi(\mathbf{q}) d\mathbf{q}. \quad (5.19)$$

Since  $A \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$ , it holds that  $A\Psi(\mathbf{q}) \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$  for a.e.  $\mathbf{q}$ . Thus, the RHS of (5.19) is positive w.r.t.  $\mathfrak{P}_M$ . This implies that  $A \otimes \mathbb{1} \geq 0$  w.r.t.  $\mathfrak{P}_M$ .

We decompose  $\Psi$  as  $\Psi = \sum_{(x,\sigma) \in \mathcal{C}_M} |x, \sigma\rangle \otimes f_{x,\sigma}$ , where  $f_{x,\sigma}(\mathbf{q}) = \langle x, \sigma | \Psi(\mathbf{q}) \rangle$ . Since  $\Psi \geq 0$  w.r.t.  $\mathfrak{P}_M$ ,  $f_{x,\sigma}(\mathbf{q}) \geq 0$  for a.e.  $\mathbf{q}$  and every  $(x, \sigma) \in \mathcal{C}_M$ . Because  $B \geq 0$  w.r.t.  $L^2(Q)_+$ , we have that  $Bf_{x,\sigma} \geq 0$  w.r.t.  $L^2(Q)_+$ , which implies  $\mathbb{1} \otimes B\Psi = \sum_{(x,\sigma) \in \mathcal{C}_M} |x, \sigma\rangle \otimes Bf_{x,\sigma} \geq 0$  w.r.t.  $\mathfrak{P}_M$ . Thus, we conclude that  $\mathbb{1} \otimes B \geq 0$  w.r.t.  $\mathfrak{P}_M$ .

Finally, we have  $A \otimes B = (A \otimes \mathbb{1})(\mathbb{1} \otimes B) \geq 0$  w.r.t.  $\mathfrak{P}_M$ .  $\square$

**Remark 5.8** Let  $A \in \mathcal{B}(\mathfrak{E}(M))$  and  $B \in \mathcal{B}(L^2(Q))$ . In this paper, we abbreviate the tensor products  $A \otimes \mathbb{1}$  and  $\mathbb{1} \otimes B$  simply as  $A$  and  $B$  if no confusion arises. For example, a claim that  $A \geq 0$  w.r.t.  $\mathfrak{P}_M$  means that  $A \otimes \mathbb{1} \geq 0$  w.r.t.  $\mathfrak{P}_M$ .  $\diamond$

**Proposition 5.9** Let  $p$  be a connector between  $(x, \sigma)$  and  $(y, \tau)$ . We have  $(-\mathbf{T})^{|p|} \geq \vartheta_{yx}\tau(p) \geq 0$  w.r.t.  $\mathfrak{P}_M$ .

*Proof.* By Lemmas 5.1, 5.5 and 5.7, we have

$$-\mathbf{T} \geq \theta_{xy}\{-T_{xy}(\sigma)\} \geq 0 \quad (5.20)$$

w.r.t.  $\mathfrak{P}_M$  for all  $x, y \in \Lambda$  and  $\sigma \in \{\uparrow, \downarrow\}$ . By using this, we have

$$(-\mathbf{T})^{\ell-1} \geq \{-\theta_{x_{\ell-1}x_\ell}T_{x_{\ell-1}x_\ell}(\sigma_\ell)\} \cdots \{-\theta_{x_1x_2}T_{x_1x_2}(\sigma_\ell)\} \quad (5.21)$$

w.r.t.  $\mathfrak{P}_M$ . Since  $\theta_{x_1x_2} \cdots \theta_{x_{\ell-1}x_\ell} = \theta_{x_1x_\ell} = \theta_{xy}$ , we obtain the desired result.  $\square$

## 5.6 Uniqueness of the ground states

The purpose of this subsection is to prove the following:

**Theorem 5.10** For each  $M \in \{-(|\Lambda| - 1)/2, -(|\Lambda| - 2)/2, \dots, (|\Lambda| - 1)/2\}$ , it holds that  $e^{-\beta H'_\infty} \geq 0$  w.r.t.  $\mathfrak{P}_M$  for all  $\beta > 0$ . Thus, the ground states of  $H'_\infty$  is unique and can be chosen to be strictly positive w.r.t.  $\mathfrak{P}_M$  by Theorem 4.9.

To prove Theorem 5.10, we begin with the following lemma.

**Lemma 5.11**  $e^{-\beta H'_\infty} \geq 0$  w.r.t.  $\mathfrak{P}_M$  for all  $\beta \geq 0$ .

*Proof.* By (5.20), we have  $(-\mathbf{T})^n \geq 0$  w.r.t.  $\mathfrak{P}_M$  for all  $n \in \mathbb{N}$ , which implies that

$$e^{-\beta \mathbf{T}} = \sum_{n=0}^{\infty} \underbrace{\frac{\beta^n}{n!}}_{\geq 0} \underbrace{(-\mathbf{T})^n}_{\geq 0} \geq 0 \quad (5.22)$$

w.r.t.  $\mathfrak{P}_M$  for all  $\beta \geq 0$ . On the other hand, we have, by Lemmas 5.4, 5.5 and 5.7,

$$e^{-\beta(U_{\text{eff}}P + \omega N_b)} = \underbrace{e^{-\beta U_{\text{eff}}P}}_{\geq 0} \underbrace{e^{-\beta \omega N_b}}_{\geq 0} \geq 0 \quad (5.23)$$

w.r.t.  $\mathfrak{P}_M$  for all  $\beta \geq 0$ . Thus, by the Trotter-Kato product formula and Proposition 4.7, we have

$$e^{-\beta H'_\infty} = \text{strong-} \lim_{n \rightarrow \infty} \left( e^{-\beta \mathbf{T}/n} e^{-\beta(U_{\text{eff}}P + \omega N_b)/n} \right)^n \geq 0 \quad (5.24)$$

w.r.t.  $\mathfrak{P}_M$  for all  $\beta \geq 0$ , because  $e^{-\beta \mathbf{T}/n} e^{-\beta(U_{\text{eff}}P + \omega N_b)/n} \geq 0$  for all  $\beta \geq 0$  and  $n \in \mathbb{N}$ .  $\square$

Let  $H'_{\infty,0}$  be the Hamiltonian  $H'_\infty$  with  $U_{xy} = 0$  for all  $x, y \in \Lambda$ .

**Proposition 5.12** *The following (i) and (ii) are equivalent:*

(i)  $e^{-\beta H'_\infty} \triangleright 0$  w.r.t.  $\mathfrak{P}_M$  for all  $\beta > 0$ .

(ii)  $e^{-\beta H'_{\infty,0}} \triangleright 0$  w.r.t.  $\mathfrak{P}_M$  for all  $\beta > 0$ .

*Proof.* To prove Proposition 5.12, it suffices to check the conditions (a)-(c) in Theorem A.1. To this end, we set  $U_{\text{eff}}^{(n)} = (1 - \frac{1}{n})U_{\text{eff}}$ . Let  $H_\infty'^{(n)} = H'_{\infty,0} + U_{\text{eff}}^{(n)}P$ . Choose  $\psi \in \mathfrak{E}[M] \hat{\otimes} \mathfrak{F}_0$ , arbitrarily, where  $\mathfrak{F}_0$  is the finite particle subspace of  $\mathfrak{F}$ <sup>5</sup> and  $\hat{\otimes}$  indicates the algebraic tensor product. We easily check that  $H_\infty'^{(n)}\psi \rightarrow H'_\infty\psi$ ,  $(H'_\infty - U_{\text{eff}}^{(n)}P)\psi \rightarrow H'_{\infty,0}\psi$  as  $n \rightarrow \infty$ . Thus, by [20, Theorem VIII. 25],  $H_\infty'^{(n)}$  converges to  $H'_\infty$  and  $H'_\infty - U_{\text{eff}}^{(n)}P$  converges to  $H'_{\infty,0}$  in the strong resolvent sense<sup>6</sup>. Thus, the condition (a) is satisfied. To check (b) is easy.

To prove (c), let  $\varphi, \psi \in \mathfrak{P}_M$  with  $\langle \varphi | \psi \rangle = 0$ . Hence,

$$\sum_{(x,\sigma) \in \mathcal{C}_M} \langle \varphi_{x,\sigma} | \psi_{x,\sigma} \rangle_{L^2(Q)} = 0. \quad (5.26)$$

Since  $\varphi_{x,\sigma}(\mathbf{q}) \geq 0$  and  $\psi_{x,\sigma}(\mathbf{q}) \geq 0$  for a.e.  $\mathbf{q}$ , it holds that  $\langle \varphi_{x,\sigma} | \psi_{x,\sigma} \rangle = 0$  for all  $(x,\sigma) \in \mathcal{C}_M$ . Recalling the notations in the proof of Lemma 5.4, we have

$$\langle \varphi | e^{-\beta U_{\text{eff}}^{(n)}P} \psi \rangle = \sum_{(x,\sigma) \in \mathcal{C}_M} e^{-\beta(1-n^{-1})U_{\text{eff}}(x,\sigma)} \langle \varphi_{x,\sigma} | \psi_{x,\sigma} \rangle = 0. \quad (5.27)$$

Thus, (c) is satisfied.  $\square$

For our purpose, it suffices to prove that  $e^{-\beta H'_{\infty,0}} \triangleright 0$  w.r.t.  $\beta > 0$  by Proposition 5.12. To this end, we use the Duhamel expansion:

$$e^{-\beta H'_{\infty,0}} = \sum_{n=0}^{\infty} D_n(\beta), \quad (5.28)$$

where

$$D_n(\beta) = \int_{0 \leq s_1 \leq \dots \leq s_n \leq \beta} \{-\mathbf{T}(s_1)\} \cdots \{-\mathbf{T}(s_n)\} e^{-\beta \omega N_b} \quad (5.29)$$

with  $\mathbf{T}(s) = e^{-s\omega N_b} \mathbf{T} e^{s\omega N_b}$ . Note that the RHS of (5.28) converges in the operator norm topology.

**Lemma 5.13** *For all  $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$  and  $\beta \geq 0$ ,  $D_n(\beta) \geq 0$  w.r.t.  $\mathfrak{P}_M$ .*

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<sup>5</sup> To be precise,

$$\mathfrak{F}_0 = \{\Phi = \{\Phi_n\}_{n=0}^\infty \mid \text{There exists an } n_0 \in \{0\} \cup \mathbb{N} \text{ such that } \Phi_n = 0 \text{ for all } n \geq n_0\}. \quad (5.25)$$

<sup>6</sup> Let  $A$  be a self-adjoint operator and let  $\{A_n\}_{n=1}^\infty$  be a family of self-adjoint operators. Then  $A_n$  is said to converge to  $A$  in the strong resolvent sense if  $(A_n - \lambda)^{-1} \rightarrow (A - \lambda)^{-1}$  for all  $\lambda$  with  $\text{Im} \lambda \neq 0$  in the strong operator topology.

*Proof.* By Lemma 5.5 and (5.20), the integrand in the RHS of (5.29) satisfies

$$\underbrace{e^{-s_1\omega N_b}}_{\geq 0} \underbrace{(-\mathbf{T})}_{\geq 0} \underbrace{e^{-(s_2-s_1)\omega N_b}}_{\geq 0} \dots \underbrace{e^{-(s_n-s_{n-1})\omega N_b}}_{\geq 0} \underbrace{(-\mathbf{T})}_{\geq 0} \underbrace{e^{-(\beta-s_n)\omega N_b}}_{\geq 0} \geq 0 \quad (5.30)$$

w.r.t.  $\mathfrak{P}_M$  for all  $0 \leq s_1 \leq \dots \leq s_n \leq \beta$ . Thus, it holds that  $D_n(\beta) \geq 0$  w.r.t.  $\mathfrak{P}_M$ .  $\square$

**Definition 5.14** We say that  $\{D_n(\beta)\}$  is *ergodic* w.r.t.  $\mathfrak{P}_M$  if, for each  $\varphi, \psi \in \mathfrak{P}_M \setminus \{0\}$ , there exist  $n \in \mathbb{N}_0$  and  $\beta \geq 0$  such that  $\langle \varphi | D_n(\beta) \psi \rangle > 0$ .  $\diamond$

**Lemma 5.15** *If  $\{D_n(\beta)\}$  is ergodic w.r.t.  $\mathfrak{P}_M$ , then  $e^{-\beta H'_{\infty,0}} \triangleright 0$  w.r.t.  $\mathfrak{P}_M$  for all  $\beta > 0$ .*

*Proof.* Choose  $\varphi, \psi \in \mathfrak{P}_M \setminus \{0\}$ , arbitrarily. By the ergodicity of  $\{D_n(\beta)\}$ , there exist  $n_0 \in \mathbb{N}_0$  and  $\beta_0 \geq 0$  such that  $\langle \varphi | D_{n_0}(\beta_0) \psi \rangle > 0$ . By Lemma 5.13, we have  $e^{-\beta_0 H'_{\infty,0}} \geq D_{n_0}(\beta_0)$  w.r.t.  $\mathfrak{P}_M$ , which implies  $\langle \varphi | e^{-\beta_0 H'_{\infty,0}} \psi \rangle \geq \langle \varphi | D_{n_0}(\beta_0) \psi \rangle > 0$ . By Theorem 4.9, we conclude that  $e^{-\beta H'_{\infty,0}} \triangleright 0$  w.r.t.  $\mathfrak{P}_M$  for all  $\beta > 0$ .  $\square$

For each  $n \in \mathbb{N}$  and  $\beta \geq 0$ , we set

$$C_n(\beta) = (-\mathbf{T})^n e^{-\beta \omega N_b}. \quad (5.31)$$

We remark that, in a way similar to Definition 5.14, the ergodicity of  $\{C_n(\beta)\}$  can be defined.

**Lemma 5.16** *If  $\{C_n(\beta)\}$  is ergodic w.r.t.  $\mathfrak{P}_M$ , then  $\{D_n(\beta)\}$  is ergodic w.r.t.  $\mathfrak{P}_M$ .*

*Proof.* We denote the integrand of the RHS of (5.29) by  $F(s_1, \dots, s_n)$ . In the proof of Lemma 5.13, we have already proved that  $F(s_1, \dots, s_n) \geq 0$  w.r.t.  $\mathfrak{P}_M$ , provided that  $0 \leq s_1 \leq \dots \leq s_n \leq \beta$ . Also remark that  $F(0, \dots, 0) = C_n(\beta)$ .

Let  $\varphi, \psi \in \mathfrak{P}_M \setminus \{0\}$ . Since  $\{C_n(\beta)\}$  is ergodic, there exist  $n_0 \in \mathbb{N}_0$  and  $\beta_0 \geq 0$  such that  $\langle \varphi | C_{n_0}(\beta_0) \psi \rangle > 0$ . Thus, since a function  $\langle \varphi | F(s_1, \dots, s_n) \psi \rangle$  is continuous in  $s_1, \dots, s_n$ , we have

$$\langle \varphi | D_{n_0}(\beta_0) \psi \rangle = \int_{0 \leq s_1 \leq \dots \leq s_{n_0} \leq \beta_0} \langle \varphi | F(s_1, \dots, s_{n_0}) \psi \rangle > 0. \quad (5.32)$$

This means that  $\{D_n(\beta)\}$  is ergodic.  $\square$

**Theorem 5.17**  *$\{C_n(\beta)\}$  is ergodic w.r.t.  $\mathfrak{P}_M$ .*

*Proof.* For each  $\varphi, \psi \in \mathfrak{P} \setminus \{0\}$ , there exist  $(x, \sigma)$  and  $(y, \tau)$  such that

$$\varphi \geq |x, \sigma\rangle \otimes \varphi_{x,\sigma}, \quad \psi \geq |y, \tau\rangle \otimes \psi_{y,\tau}, \quad (5.33)$$

where  $\varphi_{x,\sigma}, \psi_{y,\tau} \in L^2(Q)_+ \setminus \{0\}$ .<sup>7</sup> By the connectivity condition, there exists a connector  $p$  between  $(x, \sigma)$  and  $(y, \tau)$ . By Proposition 5.9, we have

$$\langle \varphi | C_{|p|}(\beta) \psi \rangle \geq \langle x, \sigma | \tau(p) | y, \tau \rangle \langle \varphi_{x,\sigma} | \theta_{xy} e^{-\beta \omega N_b} \psi_{y,\tau} \rangle. \quad (5.34)$$

<sup>7</sup> To see this, we remark that  $\varphi$  can be expressed as  $\varphi = \sum_{(x,\sigma) \in \mathcal{C}_M} |x, \sigma\rangle \otimes \varphi_{x,\sigma}$ , where  $\varphi_{x,\sigma}(\mathbf{q}) = \langle x, \sigma | \varphi(\mathbf{q}) \rangle$ . Since  $\varphi(\mathbf{q}) \geq 0$  w.r.t.  $\mathfrak{E}_+(M)$ , it holds that  $\varphi_{x,\sigma} \geq 0$  w.r.t.  $L^2(Q)_+$ . Thus,  $|x, \sigma\rangle \otimes \varphi_{x,\sigma} \in \mathfrak{P}_M$  for all  $(x, \sigma) \in \mathcal{C}_M$ , which implies  $\varphi \geq |x, \sigma\rangle \otimes \varphi_{x,\sigma}$  for all  $(x, \sigma) \in \mathcal{C}_M$ . Since  $\varphi$  is nonzero, there exists a  $(x_0, \sigma_0) \in \mathcal{C}_M$  such that  $\varphi_{x_0, \sigma_0} \neq 0$ . Hence, we obtain  $\varphi \geq |x_0, \sigma_0\rangle \otimes \varphi_{x_0, \sigma_0}$ .

Since  $\theta_{yx} \geq 0$  and  $e^{-\beta\omega N_b} \triangleright 0$  w.r.t.  $L^2(\mathcal{Q})_+$  for all  $\beta > 0$ , it holds that  $\langle \varphi_{x,\sigma} | \theta_{yx} e^{-\beta\omega N_b} \psi_{y,\tau} \rangle > 0$ , which implies that the RHS of (5.34) is strictly positive by Proposition 5.3. Thus,  $\{C_n(\beta)\}$  is ergodic.  $\square$

As a consequence of Theorem 5.17, we obtain Theorem 5.10.  $\square$

## 5.7 Completion of proof of Theorem 2.6

We introduce the spin operators by  $S^{(-)} = \sum_{x \in \Lambda} c_{x\downarrow}^* c_{x\uparrow}$ ,  $S^{(+)} = (S^{(-)})^*$ . Remark that  $S^{(-)}$  maps  $\mathfrak{E}(M)$  into  $\mathfrak{E}(M-1)$  for all  $M \in \text{spec}(S^{(3)}) \setminus \{-(|\Lambda|-1)/2\}$ . Moreover, we have the following:

**Lemma 5.18**  $S^{(-)}$  maps  $\mathfrak{E}_+(M) \setminus \{0\}$  into  $\mathfrak{E}_+(M-1) \setminus \{0\}$  for all  $M \in \text{spec}(S^{(3)}) \setminus \{-(|\Lambda|-1)/2\}$ .

*Proof.* We just remark that

$$S^{(-)}|x, \sigma\rangle = \sum_{y \in \Lambda, x \neq y} |x, t_y(\sigma)\rangle, \quad (5.35)$$

where

$$(t_y(\sigma))_z = \begin{cases} \sigma_z & z \neq y \\ \bar{\sigma}_z & z = y \end{cases} \quad (5.36)$$

with  $\bar{\uparrow} = \downarrow$  and  $\bar{\downarrow} = \uparrow$ .  $\square$

By Theorem 5.10, the ground states of  $H'_\infty$  in the  $S^{(3)} = M$  subspace is unique. We denote the unique ground state by  $\psi_M$ . Remark that  $\psi_M > 0$  w.r.t.  $\mathfrak{P}_M$ . We set  $\psi^\dagger = \psi_{M=\frac{1}{2}(|\Lambda|-1)}$ .  $\psi^\dagger$  can be expressed as

$$\psi^\dagger = \sum_{x \in \Lambda} |x, \uparrow_x\rangle \otimes f_x, \quad (5.37)$$

where  $f_x > 0$  w.r.t.  $L^2(Q)_+$  and the spin configuration  $\uparrow_x$  is defined by

$$(\uparrow_x)_y = \begin{cases} \uparrow & y \neq x \\ 0 & y = x \end{cases}. \quad (5.38)$$

By Lemma 5.18,  $(S^{(-)})^n \psi^\dagger \geq 0$  w.r.t.  $\mathfrak{P}_{M=\frac{1}{2}(|\Lambda|-n-1)}$  and  $(S^{(-)})^n \psi^\dagger \neq 0$  whenever  $0 \leq n \leq 2(|\Lambda|-1)$ . Let  $E_M$  be the ground state energy of  $H'_\infty$  in the  $S^{(3)} = M$  subspace. We set  $E^\dagger = E_{M=\frac{1}{2}(|\Lambda|-1)}$ . Since  $S^{(-)}$  commutes with  $H'_\infty$ , we have  $H'_\infty (S^{(-)})^n \psi^\dagger = E^\dagger (S^{(-)})^n \psi^\dagger$ . Because  $\psi_M$  is strictly positive w.r.t.  $\mathfrak{P}_M$ , we have  $\langle \psi_M | (S^{(-)})^{|\Lambda|-2M-1} \psi^\dagger \rangle > 0$ . Hence,  $(S^{(-)})^{|\Lambda|-2M-1} \psi^\dagger$  is the ground state of  $H'_\infty$  in the  $S^{(3)} = M$  subspace and  $E^\dagger = E_M$  for all  $M \in \text{spec}(S^{(3)})$ .  $\square$



## 6 Proof of Theorems 2.8 and 2.9

### 6.1 A remark on the Hamiltonian

We have to clarify a mathematical definition of the integral  $\int_{C_{xy}} dr \cdot A(r)$ . For each  $x, y \in \Lambda$  with  $x \neq y$ , let

$$F_{x,y}(k) = \frac{1}{ik \cdot (y - x)} (e^{ik \cdot y} - e^{ik \cdot x}). \quad (6.1)$$

Remark that  $|F_{x,y}(k)| \leq 1$  for every  $k \in V^* \setminus \{0\}$ . We define a field operator  $\phi$  by

$$\phi = \sum_{k \in V^*} \sum_{\lambda=1,2} \frac{\chi_\kappa(k)}{\sqrt{\omega(k)}} \varepsilon(k, \lambda) \cdot \frac{x - y}{|x - y|} \left\{ F_{x,y}(k) a(k, \lambda) + \text{h.c.} \right\}. \quad (6.2)$$

$\phi$  is essentially self-adjoint on  $\mathfrak{E}(M) \hat{\otimes} \mathfrak{R}_0$ . Here,  $\mathfrak{R}_0$  is the finite particle subspace.

<sup>8</sup> By choosing  $C_{xy}$  as  $C_{xy} = \{(1 - s)x + sy \in V \mid s \in [0, 1]\}$ , we easily check that  $\int_{C_{xy}} dr \cdot A(r) = \phi$  on  $\mathfrak{E}(M) \hat{\otimes} \mathfrak{R}_0$ . Thus,  $\int_{C_{xy}} dr \cdot A(r)$  is essentially self-adjoint on  $\mathfrak{E}(M) \hat{\otimes} \mathfrak{R}_0$ . We denote its closure by the same symbol.

### 6.2 Sketch of the proof

Since the proof of Theorem 2.8 is similar to that of Theorem 2.5, we omit it. We provide a sketch of a proof of Theorem 2.9.

We switch to the  $\mathcal{Q}$ -representation [7]. In the  $\mathcal{Q}$ -representation,  $\mathfrak{R}$  is identified with  $L^2(\mathcal{Q}, d\mu)$ , where  $d\mu$  is some Gaussian measure.  $A(x)$  can be regarded as a multiplication operator on  $L^2(\mathcal{Q}, d\mu)$  for all  $x \in V$ .

For notational convenience, we introduce two operators:

$$H_f = \sum_{k \in V^*} \sum_{\lambda=1,2} \omega(k) a(k, \lambda)^* a(k, \lambda), \quad N_f = \sum_{k \in V^*} \sum_{\lambda=1,2} a(k, \lambda)^* a(k, \lambda). \quad (6.4)$$

$H_f$  and  $N_f$  are essentially self-adjoint. So, we denote their closures by the same symbols.

Let  $\xi = e^{i\pi N_f/2}$ . Let  $E(x) = \xi A(x) \xi^{-1}$ . Then we have the following:

**Lemma 6.1** (i)  $e^{i\mathbf{a} \cdot E(x)} \geq 0$  w.r.t.  $L^2(\mathcal{Q}, d\mu)_+$  for all  $\mathbf{a} \in \mathbb{R}^3$  and  $x \in V$ .

(ii)  $e^{-\beta H_f} \triangleright 0$  w.r.t.  $L^2(\mathcal{Q}, d\mu)_+$  for all  $\beta > 0$ .

*Proof.* Proofs of (i) and (ii) are similar to those of [7, Lemma 7.27] and [7, Proposition 7.28], respectively.  $\square$

For each  $x, y \in \Lambda$ , we define

$$\Phi_{xy} = \exp \left\{ i \int_{C_{xy}} dr \cdot E(r) \right\}. \quad (6.5)$$

---

<sup>8</sup> To be precise,

$$\mathfrak{R}_0 = \left\{ \Phi = \{\Phi_n\}_{n=0}^\infty \in \mathfrak{R} \mid \text{There exists an } n_0 \in \mathbb{N}_0 \text{ such that, if } n \geq n_0, \text{ then } \Phi_n = 0 \right\}. \quad (6.3)$$

**Lemma 6.2** For each  $x, y \in \Lambda$ , we have  $\Phi_{xy} \succeq 0$  w.r.t.  $L^2(\mathcal{Q}, d\mu)_+$ .

*Proof.* For each  $N \in \mathbb{N}$ , we set

$$\mathcal{A}_N^{x,y} = \sum_{j=1}^{N+1} \frac{1}{N} \frac{x-y}{|x-y|} \cdot A\left(x + \frac{j-1}{N}(y-x)\right). \quad (6.6)$$

$\mathcal{A}_N^{x,y}$  is essentially self-adjoint. We denote its closure by the same symbol. Using the formulas  $\|(e^{iB} - 1)\varphi\| \leq \|B\varphi\|$  with  $B$  self-adjoint and  $\|a(f)^\# \varphi\| \leq \|f\| \|(N_f + \mathbb{1})\varphi\|$ , we have

$$\left\| \left( e^{i \int_{C_{xy}} dr \cdot A(r)} - e^{i \mathcal{A}_N^{x,y}} \right) \varphi \right\| \leq \left\| \frac{\chi_\kappa}{\sqrt{\omega}} \varepsilon \cdot \frac{x-y}{|x-y|} (F_{x,y} - F_{x,y}^N) \right\| \|(N_f + \mathbb{1})\varphi\| \quad (6.7)$$

for all  $\varphi \in \mathfrak{E}(M) \hat{\otimes} \mathfrak{R}_0$ , where  $F_{x,y}^N(k) = \sum_{j=1}^{N+1} \frac{1}{N} \exp\{ik \cdot (x + \frac{j-1}{N}(y-x))\}$ . Since  $F_{x,y}^N$  converges to  $F_{x,y}$  in  $\ell^2(V^* \times \{1, 2\})$ ,  $e^{i \mathcal{A}_N^{x,y}}$  strongly converges to  $e^{i \int_{C_{xy}} dr \cdot A(r)}$  on  $\mathfrak{E}(M) \hat{\otimes} \mathfrak{R}_0$  by (6.7). Since  $\mathfrak{E}(M) \hat{\otimes} \mathfrak{R}_0$  is dense in  $\mathfrak{E}(M) \otimes \mathfrak{R}$ , the convergence holds on whole  $\mathfrak{E}(M) \otimes \mathfrak{R}$ .

Since

$$\xi e^{i \mathcal{A}_N^{x,y}} \xi^{-1} = \prod_{j=1}^{N+1} \exp \left\{ i \frac{1}{N} \frac{x-y}{|x-y|} \cdot E\left(x + \frac{j-1}{N}(y-x)\right) \right\}, \quad (6.8)$$

we have, by Lemma 6.1 (i),  $\xi e^{i \mathcal{A}_N^{x,y}} \xi^{-1} \succeq 0$  w.r.t.  $L^2(\mathcal{Q}, d\mu)_+$ . Because  $\xi e^{i \mathcal{A}_N^{x,y}} \xi^{-1}$  strongly converges to  $e^{i \int_{C_{xy}} dr \cdot E(r)}$ , we conclude the desired result by Proposition 4.7.  $\square$

Let

$$\mathbb{T} = \sum_{x,y \in \Lambda} \sum_{\sigma=\uparrow, \downarrow} \Phi_{xy} T_{xy}(\sigma). \quad (6.9)$$

We define a self-dual cone  $\mathfrak{D}_M$  by

$$\mathfrak{D}_M = \left\{ \Psi \in \mathfrak{E}(M) \otimes L^2(\mathcal{Q}, d\mu) \mid \Psi(\phi) \geq 0 \text{ w.r.t. } \mathfrak{E}_+(M) \text{ for } \mu\text{-a.e.} \right\}. \quad (6.10)$$

Here, we use the following identification:

$$\mathfrak{E}(M) \otimes L^2(\mathcal{Q}, d\mu) = \int_{\mathcal{Q}}^{\oplus} \mathfrak{E}(M) d\mu. \quad (6.11)$$

Corresponding to Proposition 5.9, we have the following:

**Proposition 6.3** Let  $p$  be a connector between  $(x, \sigma)$  and  $(y, \tau)$ . We have  $(-\mathbb{T})^{|p|} \succeq \Phi_{xy} \tau(p) \succeq 0$  w.r.t.  $\mathfrak{D}_M$ .

Let

$$\mathbb{C}_n(\beta) = (-\mathbb{T})^n e^{-\beta H_f}. \quad (6.12)$$

In a similar way as in the proof of Theorem 5.10, we can prove the following:

**Lemma 6.4** *If  $\{\mathbb{C}_n(\beta)\}$  is ergodic w.r.t.  $\mathfrak{D}_M$ , then  $\xi e^{-\beta H_\infty} \xi^{-1} \triangleright 0$  w.r.t.  $\mathfrak{D}_M$  for all  $\beta > 0$ .*

Thus, it suffices to prove that  $\{\mathbb{C}_n(\beta)\}$  is ergodic w.r.t.  $\mathfrak{D}_M$ .

For each  $\varphi, \psi \in \mathfrak{D}_M \setminus \{0\}$ , there exist  $(x, \sigma)$  and  $(y, \tau)$  such that

$$\varphi \geq |x, \sigma\rangle \otimes \varphi_{x, \sigma}, \quad \psi \geq |y, \tau\rangle \otimes \psi_{y, \tau}, \quad (6.13)$$

where  $\varphi_{x, \sigma}, \psi_{y, \tau} \in L^2(\mathcal{Q}, d\mu)_+ \setminus \{0\}$ . By the connectivity condition, there exists a connector  $p$  between  $(x, \sigma)$  and  $(y, \tau)$ . By Proposition 6.3, we have

$$\langle \varphi | \mathbb{C}_{|p|}(\beta) \psi \rangle \geq \langle x, \sigma | \tau(p) | y, \tau \rangle \langle \varphi_{x, \sigma} | \Phi_{xy} e^{-\beta H_f} \psi_{y, \tau} \rangle. \quad (6.14)$$

Since  $\Phi_{xy} \geq 0$  and  $e^{-\beta H_f} \triangleright 0$  w.r.t.  $L^2(\mathcal{Q}, d\mu)_+$  by Lemma 6.1, we obtain  $\langle \varphi_{x, \sigma} | \Phi_{xy} e^{-\beta H_f} \psi_{y, \tau} \rangle > 0$ , which implies that the RHS of (6.14) is strictly positive by Proposition 5.3.  $\square$

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## A A fundamental theorem

**Theorem A.1** *Let  $A$  and  $B$  be self-adjoint operators, bounded from below. Assume the following conditions:*

- (a) *There exists a sequence of bounded self-adjoint operator  $C_n$  such that  $A + C_n$  converges to  $B$  in the strong resolvent sense and  $B - C_n$  converges to  $A$  in the strong resolvent sense as  $n \rightarrow \infty$ ;*
- (b)  *$e^{-tC_n} \geq 0$  w.r.t.  $\mathfrak{P}$  for all  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$ ;*
- (c) *For all  $\xi, \eta \in \mathfrak{P}$  such that  $\langle \xi | \eta \rangle = 0$ , it holds that  $\langle \xi | e^{-tC_n} \eta \rangle = 0$  for all  $n \in \mathbb{N}$  and  $t \geq 0$ .*

*The following (i) and (ii) are mutually equivalent:*

- (i)  *$e^{-tA} \triangleright 0$  w.r.t.  $\mathfrak{P}$  for all  $t > 0$ ;*
- (ii)  *$e^{-tB} \triangleright 0$  w.r.t.  $\mathfrak{P}$  for all  $t > 0$ .*

*Proof.* The proof is similar to [1, Theorem 3]. For readers' convenience, we provide a proof.

(i)  $\implies$  (ii): By (a) and the Trotter-Kato formula, we have

$$e^{-tB} = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \left( e^{-tA/k} e^{-tC_n/k} \right)^k \quad (A.1)$$

in the strong operator topology. Since  $e^{-tA} \geq 0$  w.r.t.  $\mathfrak{P}$  for all  $t \geq 0$  and (b), we conclude that  $e^{-tB} \geq 0$  w.r.t.  $\mathfrak{P}$  for all  $t \geq 0$  by Proposition 4.7.

Let  $\xi \in \mathfrak{P} \setminus \{0\}$ . We set  $K(\xi) = \{\eta \in \mathfrak{P} | \langle \eta | e^{-tB} \xi \rangle = 0 \ \forall t \geq 0\}$ . To prove (ii), it suffices to show that  $K(\xi) = \{0\}$ . If  $\eta \in \mathfrak{P}$  and  $\langle \eta | e^{-tB} \xi \rangle = 0$ , then, by (c),  $\langle e^{sC_n} \eta | e^{-tB} \xi \rangle = 0$ . Hence,  $e^{sC_n} K(\xi) \subseteq K(\xi)$  for all  $s \in \mathbb{R}$ . It is trivial that  $e^{-tB} K(\xi) \subseteq K(\xi)$  for all  $t \geq 0$ . Accordingly,  $(e^{-tB/k} e^{tC_n/k})^k K(\xi) \subseteq K(\xi)$ , which implies  $e^{-tA} K(\xi) \subseteq K(\xi)$  by the Trotter-Kato product formula. In particular, if  $\eta \in K(\xi)$ , then  $\langle \eta | e^{-tA} \xi \rangle = 0$ . Since  $e^{-tA} \triangleright 0$  w.r.t.  $\mathfrak{P}$  for all  $t > 0$ , we conclude that  $\eta = 0$ .

Similarly, we can prove that (ii)  $\implies$  (i).  $\square$

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